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DECOMPOSITION IN THE DYNAMIC PROBLEMS WITH CONTROL

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A nonlinear problem of optimal control is decomposed into problems of smaller dimension. The initial problem describes the dynamics of the system in the case when the constraints corresponding to the subsystems are separated out, and the general type constraints are present. Criteria of optimality of the admissible intermediate solutions are established together with the monotonus character of the iterative process with respect to the functional of the initial problem.

We consider the problem of optimal control of the form [1, 2]

$$dx_j(t)/dt = A_j(t)\,x_j(t) + b_j(u_j(t), t), \quad u_j(t) \geqslant 0 \tag{1}$$

$$x_j(0) = \varkappa_j, \ v_j(x_j(T)) \leqslant 0, \ p_j(x_j(t), u_j(t), t) \leqslant 0; \ j \in J \tag{2}$$

$$\sum_{j=1}^{J} q_i(x_j(T)) \leqslant 0, \sum_{j=1}^{J} d_j(x_j(t), u_j(t)) \leqslant 0 \tag{3}$$

$$F(u_j) = \sum_{j=1}^{J} w(x_j(T)) + \int_0^T \sum_{j=1}^{J} c_j(x_j(t), u_j(t), t)\,dt \to \max \tag{4}$$

The relations (1) — (3) describe the dynamics of the controlled system consisting of J subsystems, a fixed index $j = 1, \ldots, J$ corresponding to each subsystem. For every $j \in J$ the dimensions of the vector functions $\varkappa_j$, b_j and x_j are N_j, u_j — I, v_j — S_j, q_j — R, p_j — K_j, d_j — L, and the dimension of the matrix A_j is $N_j \times N_j$. The quantities u_j represent the controls, x_j are the phase variables and t is time. The actual meaning of the remaining variables in (1) — (4) is given in e.g. [2].

The constraints (1) and inequalities (1), (2) represent the restrictions for the subsystems, and the inequalities (3) represent the general constraints of the dynamic system and are separable, together with the functional (4), over the subsystems. Concrete forms of the subsystems can be interpreted with help of models [2].

The problem (1) — (4) represents a Bolza-type problem [3] of optimal control. The equalities in (2) represent the initial conditions, while the second and first inequalities in (2) and (3) respectively represent the conditions at the right end. The last relations in (2) and (3) represent mixed constraints. The problem in question consists, generally speaking, of determining bounded variables on the interval $[0, T]$ ensuring that (4) attains its maximum under the constraints (1) — (3).

The initial problem has the following dimension with respect to the controls and phase variables

$$I \times J + \sum_{j=1}^{J} N_j$$

The aim of this paper is to reduce the problem (1) — (4) to problems of smaller dimension. In [4] such a decomposition is carried out for the linear problems of optimal control.

We assume that the following conditions [1, 2] hold for the functions appearing in the problem. The functions $-p_j(x_j, u_j, t)$, $-d_j(x_j, u_j, t)$, $b_j(u_j, t)$, $c_j(x_j, u_j, t)$ are continuously differentiable over the whole space, are concave in x_j and u_j, and increase monotonously in x_j; functions $-v_j(\beta_j)$, $-q_j(\beta_j)$, $w_j(\beta_j)$ are continuously differentiable, concave, and increase monotonously in β_j. We also assume that the measurable bounded components of the matrices $A_j(t)$ are greater than, or equal to zero almost everywhere on the interval $[0, T]$.

Let us assume that conditions [1, 2] ensuring the reduction of the problems of optimal control to the problems of convex programming in Banach spaces, for which the action principle is valid, hold for the initial problem and for the intermediate problems discussed below. These conditions reduce, in particular, to fulfilling the Slater condition with respect to the constraints and inequalities

$$x_j(t) \geqslant \varepsilon > 0, \quad p_j(0, 0, t) \leqslant 0, \quad \sum_{j=1}^{J} d_j(0, 0, t) \leqslant 0$$

The decomposition is constructed according to the scheme described in [4]. We introduce the macrocontrols $U^i(t)$ and functions $\alpha_j^i(t)$

$$U^i(t) = \sum_{j=1}^{J} u_j^i(t), \quad \alpha_j^i(t) = \frac{u_j^i(t)}{U^i(t)}, \quad i \in I$$

Assuming $\alpha_j^i(t)$ to be fixed, we obtain from (1) — (4) the following problem with macrocontrol:

$$\begin{gathered}
\frac{dx_j(t)}{dt} A_j(t)\, x_j(t) + B_j(U^i(t), t), \quad U^i(t) \geqslant 0 \\
x_j(0) = \varkappa_j, \; v_j(x_j(T)) \leqslant 0, \; P_j(x_j(t), U^i(t), t) \leqslant 0 \\
\sum_{j=1}^{J} q_j(x_j(T)) \leqslant 0, \quad D(x_j(t), U^i(t), t) \leqslant 0 \\
g(U^i) = \sum_{j=1}^{J} w_j(x_j(T)) + \int_0^T C(x_j(t), U^i(t), t)\, dt \to \max \\
B_j(U^i, t) = b_j(\alpha_j^i U^i, t), \; P_j(x_j, U^i, t) = p_j(x_j, \alpha_j^i U^i, t) \\
D(x_j, U^i, t) = \sum_{j=1}^{J} d_j(x_j, \alpha_j^i U^i, t), \; C(x_j, U^i, t) = \sum_{j=1}^{J} c_j(x_j, \alpha_j^i U^i, t)
\end{gathered} \tag{5}$$

Let us consider, for (5), a problem which is dual in the Wolf sence [5]

$$-\frac{d\chi_j(t)}{dt} = A_j^*(t)\, \chi_j(t) - \frac{\partial P_j(x_j, U^i, t)}{\partial x_j} \eta_j(t) - \frac{\partial D(x_j, U^i, t)}{\partial x_j} \delta(t) + \frac{\partial C(x_j, U^i, t)}{\partial x_j} \tag{6}$$

$$-\sum_{j=1}^{J}\frac{\partial B_j(U^i,t)}{\partial U^i}\chi_j(t)+\sum_{j=1}^{J}\frac{\partial P_j(x_j,U^i,t)}{\partial U^i}\eta_j(t)+$$
$$\frac{\partial D(x_j,U^i,t)}{\partial U^i}\delta(t)\geqslant\frac{\partial C(x_j,U^i,t)}{\partial U^i}$$
$$\chi_j(T)=-\frac{\partial v_j(x_j(T))}{\partial x_j}\omega_j-\frac{\partial q_j(x_j(T))}{\partial x_j}\nu+\frac{\partial w_j(x_j(T))}{\partial x_j}$$
$$\eta_j(t)\geqslant 0,\ \delta(t)\geqslant 0,\ \omega_j\geqslant 0,\ \nu\geqslant 0$$
$$\psi=\sum_{j=1}^{J}w_j(x_j(T))+\int_0^T\left\{C(x_j,U^i,t)+\sum_{j=1}^{J}\Omega_j(t)\right\}dt\to\min$$
$$\Omega_j=\Omega_{1j}-\Omega_{2j}$$
$$\Omega_{j1}=[B_j(U^i,t)+A_j(t)x_j(t)-dx_j(t)/dt]\,\chi_j(t)$$
$$\Omega_{j2}=P_j(x_j,U^i,t)\,\eta_j(t)+D(x_j,U^i,t)\,\delta(t)+v_j(x_j(T))\,\omega_j+q_j\times(x_j(T))\,\nu$$

Here $A_j^*(t)$ denote the matrices obtained by transposing $A_j(t)$; the variables $\chi_j(t)$ represent dual impulses corresponding to the differential constraints (5). The dual variables $\omega_j, \eta_j(t), \nu, \delta(t)$ satisfy the last inequalities of (5). The dimensions of the vector functions $\chi_j(t)$, $\eta_j(t)$, $\delta(t)$ are, respectively, N_j, K_j and L, and the dimensions of the vectors ω_j and ν are S_j and R.

Let unique extremal solutions $U^{i0}(t)>0$, $x_j^\circ(t)$ and $\chi_j^\circ(t)$, $\eta_j^\circ(t)$, $\delta^\circ(t)$, ω_j°, ν° be found for the pair of conjugate problems (5) and (6), for some fixed values of $\alpha_j^i(t)$. The problems for the subsystems decompose into J problems where for every $j\in J$ we have the constraints (1) and restrictions (2), and the functionals (4) are supplemented by the terms

$$-q_j(x_j(T))\,\nu^0-\int_0^T d_j(x_j(t),u_j(t),t)\,\delta^0(t)\,dt$$

Let $u_j^*(t)$ be optimal bounded solutions of the problems for the subsystems and $u_j^{i0}(t)$ be defined by the relation $u_j^{i0}(t)=\alpha_j^i(t)\,U^{i0}(t)$. We shall call the quantities $u_j^{i0}(t)$ the deaggregated controls. We introduce the function $\alpha_j^i(t,\rho_j)$ in accordance with the relation

$$\alpha_j^i(t,\rho_j)=[u_j^{i0}(t)+\rho_j(u_j^{i*}(t)-u_j^{i0}(t))]/Z \tag{7}$$
$$Z=\sum_{j=1}^{J}[u_j^{i0}(t)+\rho_j(u_j^{i*}(t)-u_j^{i0}(t))]$$

where the parameters ρ_j accompanying every $j\in J$ belong to the segment $[0, 1]$.

Substituting $\alpha_j^i(t,\rho_j)$ given by (7) into the initial problem (1) — (4), we obtain a problem with macrocontrols depending on the parameters ρ_j. Let us denote by $g^\circ(\rho_j)$ the extremal value of the functional in (5) as a function of ρ_j. Let the maximum of the function $g^\circ(\rho_j)$ on a unit cube be attained at ρ_j°. Then the functions $\alpha_j^i(t)$ for use in the next step of the iterative process can be obtained by substituting ρ_j° into (7).

Thus the initial problem (1) — (4) is reduced to problems of smaller dimension.

The number of the unknowns in the problem with macrocontrols is given by

$$I + \sum_{j=1}^{J} N_j$$

Every problem for a subsystem has $I + N_j$ variables, and the problem of determining the maximum of $g^\circ(\rho)$ includes J_j variables.

Next we construct a sequence of controls $u_j^\circ(t)$ and solutions $x_j^\circ(t)$ admissible by the initial problem (1) — (4). We formulate the criteria of optimality (condition of the termination of the process) and establish the monotonous character of the iterative process in terms of the functional.

Let $x_j^*(t)$ denote the extremal solutions of the problems for the subsystems corresponding to the controls $u_j^*(t)$. Then the condition that the solution $u_j^\circ(t)$, $x_j^\circ(t)$ of the problem (1) — (4) is extremal, consists of satisfying the equation

$$\sum_{j=1}^{J} \Lambda_j + \int_0^T \left\{\sum_{j=1}^{J} [c_j(x_j^*, u_j^*, t) - c_j(x_j^\circ, u_j^\circ, t)]\right\} dt = 0 \tag{8}$$

$$\Lambda_j = w_j(x_j^*(T)) - q_j(x_j^*(T))\,\nu^\circ - w_j(x_j^\circ(T)) - \int_0^T d_j(x_j^*, u_j^*, t)\,\delta^\circ(t)\,dt$$

The relation (8) is derived in the manner analogous to that in ([4]. Let $\lambda_j^*(t)$, $\zeta_j^*(t), \gamma_j^*$ be the extremal solutions of the problems which are duals of the problems for the subsystems. Then the set $\lambda_j^*(t)$, $\zeta_j^*(t), \delta^\circ(t), \gamma_j^*, \nu^\circ$ and $u_j^*(t), x_j^*(t)$ will be admissible for the problem dual to the initial problem (1) — (4). We have the following inequality in the values of the functionals for the admissible solutions of a pair of conjugated problems

$$\sum_{j=1}^{J} \left(w_j^* + \int_0^T c_j^*(t)\,dt\right) \geqslant \sum_{j=1}^{J} \left(w_j^\circ + \int_0^T c_j^\circ(t)\,dt\right) \tag{9}$$

$$w_j^* = w_j(x_j^*(T)) - v_j(x_j(T))\,\gamma^* - q_j(x_j^*(T))\,\nu^\circ, \quad w_j^\circ = w_j(x_j^\circ(T))$$
$$c_j^*(t) = c_j(x_j^*, u_j^*, t) + b_j^*(t)\,\lambda_j^*(t) + p_j(x_j^*, u_j^*, t)\,\zeta_j^*(t) + d_j(x_j^*, u_j^*, t)\,\delta^\circ(t)$$
$$b_j^*(t) = b_j(u_j^*, t) - A_j(t)\,x_j^*(t) - dx_j^*(t)/dt$$
$$c_j^\circ(t) = c_j(x_j^\circ, u_j^\circ, t)$$

Terms containing $b_j^*(t)$ in the left hand side integrand in (9) are equal to zero by virtue of (1). In addition we have

$$\int_0^T \left[\sum_{j=1}^{J} p_j(x_j^*, u_j^*, t)\,\zeta_j^*(t)\right] dt = 0, \quad \sum_{j=1}^{J} v_j(x_j^*(T))\,\gamma_j^* = 0$$

by virtue of the condition [2] of complementary flexibility.

Thus (8) follows from (9). The equations in (9) and (8) ensure the extremality of the solution $u_j^\circ(t), x_j^\circ(t)$ of the problem (1) — (4). If $u_j^\circ(t), x_j^\circ(t)$ are not extremal for the initial problem, then strict inequality obtains in (8).

We shall show now that the iterative process in terms of the functional is strictly monotonous. Let us assume that a solution $U^{i0}(t) > 0$ of the problem (5) with macrocontrols is obtained for some $\alpha_j^i(t)$ and, that the corresponding solution

$u_j^\circ(t)$, $x_j^\circ(t)$ is not optimal for the initial problem. Let us consider the functions $\alpha_j^i(t, \rho)$ given by (7) where $\rho_j = \rho$. We obtain the derivative with respect to ρ of the value of the functional $g^\circ(\rho)$ at the point $\rho = 0$, and denote it by $(g^\circ(0))'$. This derivative is calculated, as in [4], like the derivative of the Lagrange function of the problem (5), i.e. of the functional in (6), with $\partial\alpha_j^i(t, 0) / \partial\rho$ obtained from (7), taken into account.

Paying due regard to the dependence of $x_j^\circ(t)$ on ρ, we shall consider, by virtue of the assumption of uniqueness, their extremal values as the dual variables. Identity transformations and use of (5) yield finally

$$(g^\circ(0))' = \int_0^T \sum_{j=1}^J \left(\frac{\partial c_j}{\partial u_j} u_j^* + \frac{\partial b_j}{\partial u_j} u_j^* \chi_j^\circ - \frac{\partial p_j}{\partial u_j} u_j^* \eta_j^\circ - \frac{\partial d_j}{\partial u_j} u_j^* \delta^\circ\right) dt \tag{10}$$

where the partial derivatives are computed from the solution $u_j^\circ(t)$, $x_j^\circ(t)$.

Multiplying the second relations of (6) by $U^{i0}(t)$, summing over i and integrating, we obtain from (6) and (10)

$$(g^\circ(0))' = \int_0^T \sum_{j=1}^J \left[\frac{\partial c_j}{\partial u_j}(u_j^* - u_j^\circ) + \frac{\partial b_j}{\partial u_j}(u_j^* - u_j^\circ)\chi_j^\circ - \frac{\partial p_j}{\partial u_j}(u_j^* - u_j^\circ)\eta_j^\circ - \frac{\partial d_j}{\partial u_j}(u_j^* - u_j^\circ)\delta^\circ\right] dt \tag{11}$$

Multiplying the first relation of (6) by $(x_j^* - x_j^\circ)$, summing over $n \in N_j$ and over j, integrating, with (5) taken into account, and paying due regard to the initial conditions (2), we obtain

$$\sum_{j=1}^J \left\{\int_0^T [(x_j^* - x_j^\circ) H_j + \varphi_j]\, dt + (x_j^*(T) - x_j^\circ(T))\chi_j^\circ(T) = 0\right. \tag{12}$$

$$H_j = \partial c_j / \partial x_j - \eta_j^\circ \partial p_j / \partial x_j - \delta^\circ \partial d_j / \partial x_j$$

$$\varphi_j = [b_j(u_j^\circ, t) - b_j(u_j^*, t)]\chi_j^\circ$$

Next we take (11) and (12), replace $\chi_j^\circ(T)$ by their expressions given in (6) and add to (11) the expressions

$$\int_0^T \sum_{j=1}^J p_j(x_j^\circ, u_j^\circ, t)\eta_j^\circ dt, \quad \int_0^T \sum_{j=1}^J d_j(x_j^\circ, u_j^\circ, t)\delta^\circ dt$$

$$\sum_{j=1}^J v_j(x_j^\circ(T))\omega_j^\circ, \quad \sum_{j=1}^J q_j(x_j^\circ(T))\nu^\circ$$

which are all equal to zero by virtue of the conditions of complementary flexibility for the problem (5). This yields the following final expression for the derivative $(g^\circ(0))'$:

$$(g^\circ(0))' = \pi_1 + \pi_2 + \pi_3 + \pi_4 \tag{13}$$

$$\pi_1 = \sum_{j=1}^{J} \Lambda_j + \int_0^T \left\{ \sum_{j=1}^{J} [c_j(x_j^*, u_j^*, t) - c_j(x_j^\circ, u_j^\circ, t)] \right\} dt$$

$$\pi_2 = \int_0^T \left\{ \sum_{j=1}^{J} M_{xu} [(c_j, 1) + (b_j, \chi_j^\circ) - (p_j, \eta_j^\circ) - (d_j, \delta^\circ)] + \right.$$

$$\left. M_x [(w_j, 1) - (v_j, \omega_j^\circ) - (q_j, \nu^\circ)] \right\} dt$$

$$\pi_3 = -\int_0^T \sum_{j=1}^{J} p_j(x_j^*, u_j^*, t)\, \eta_j^\circ\, dt, \quad \pi_4 = -\sum_{j=1}^{J} v_j(x_j^*(T))\, \omega_j^\circ$$

$$M_{xu}(z, \mu) = \left(z(x_j^\circ, u_j^\circ, t) + \frac{\partial z}{\partial x_j}(x_j^* - x_j^\circ) + \right.$$

$$\left. \frac{\partial z}{\partial u_j}(u_j^* - u_j^\circ) - z(x_j^*, u_j^*, t) \right) \mu$$

$$M_x(z, \mu) = \left(z(x_j^*(T)) - \frac{\partial z}{\partial x_j}(x_j^*(T) - x_j^\circ(T)) - z(x_j^\circ(T)) \right) \mu$$

Here $\pi_1 > 0$ since it is the left side of (8); $\pi_2 \geqslant 0$ by virtue of (6), of the assumption that the functions appearing in the expression are convex and of the inequality $\chi_j(t) \geqslant 0$ used for solving the equations (6) with a condition given at the right end. Moreover we have $\chi_j(T) \geqslant 0$ and $d\chi_j(t)/dt \leqslant 0$ in accordance with the assumption that the functions appearing in the expression are monotonous and the inequality $A_j(t) \geqslant 0$; $\pi_3 \geqslant 0$, $\pi_4 \geqslant 0$ by virtue of the relations (1) and (6).

From (13) it follows that $g^\circ(\rho) \dot{>} g^\circ(0)$ in some neighborhood of the point $\rho = 0$, and this implies that the iterative process in terms of the functional is monotonous.

The local monotonousness is shown by the assumption that the extremal dual variables of the problem (6) are unique. The case of non-uniqueness is investigated according to the scheme given in [4]. The monotony and the relation (13) together imply, as in [4], that the solution of the problem (1) — (4) tends to its extremal value.

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